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$O(2)$-symmetry breaking bifurcation: with application to the flow past a sphere in a pipe

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SUMMARY

The steady, axisymmetric laminar flow of a Newtonian fluid past a centrally-located sphere in a pipe first loses stability with increasing flow rate at a steady $O(2)$-symmetry breaking bifurcation point. Using group theoretic results, a number of authors have suggested techniques for locating singularities in branches of solutions that are invariant with respect to the symmetries of an arbitrary group. These arguments are presented for the $O(2)$-symmetry encountered here and their implementation for $O(2)$-symmetric problems is discussed. In particular, how a bifurcation point may first be detected and then accurately located using an 'extended system' is described. Also shown is how to decide numerically if the bifurcating branch is subcritical or supercritical. The numerical solutions were obtained using the finite element code ENTWIFE. This has enabled the computation of the symmetry breaking bifurcation point for a range of sphere-to-pipe diameter ratios. A wire along the centerline of the pipe downstream of the sphere is also introduced, and its effect on the critical Reynolds number is shown to be small. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: flow past a sphere; numerical bifurcation; $O(2)$-symmetry breaking

1. INTRODUCTION

The external flow past a sphere has attracted considerable attention for at least the past 60 years, since the pioneering experimental work of Möller [1]. More recent experimental investigations by Goldburg and Florsheim [2], Margarvey and Bishop [3], Nakamura [4] and Wu and Faeth [5], and computational studies by Natarajan and Acrivos [6], Tomboulides *et al.* [7] and Tavener [8], have all concluded that the initially steady axisymmetric flow past the sphere loses stability to a steady, asymmetric flow above a critical flow rate. The experimental

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studies and the computations of Tomboulides *et al.* [7] have shown that this steady asymmetric flow in turn loses stability to a time-dependent flow with a further small increase in the flow rate.

Natarajan and Acrivos [6] and Tavener [8] both used the finite element method (FEM) to compute the steady axisymmetric flow past a sphere. Natarajan and Acrivos [6] investigated the classical exterior problem in an unbounded domain, while Tavener [8] located the sphere along the axis of a pipe and thereby introduced a new parameter, the 'blockage ratio', which is the ratio of sphere diameter to pipe diameter. The basic axisymmetric flow is linearly stable if and only if all eigenvalues of a large generalized non-symmetric eigenvalue problem lie in the stable half plane, but the flow loses stability as soon as a single eigenvalue or eigenvalue pair crosses the imaginary axis into the unstable half plane. Natarajan and Acrivos [6] employed a shift and invert strategy on a complex matrix eigenvalue problem, while Tavener [8] used a Cayley transform technique [9] on an equivalent real matrix eigenvalue problem of the same dimension. In both the unbounded domain and in the pipe for blockage ratios between 0.2 and 0.7, a real eigenvalue corresponding to an eigenvector with an azimuthal wavenumber of 1, was determined to be the first to cross the imaginary axis, indicating the loss of stability to a steady asymmetric flow with azimuthal wavenumber of 1.

Relatively few numerical studies have investigated the role played by the non-slip lateral boundaries that are obtained when the sphere is located in a tube. Johansson [10] computed the axisymmetric flow past a sphere in a cylinder at a single blockage ratio of 0.1, and more recently Wham *et al.* [11] have calculated the flow for a range of blockage ratios. Bozzi *et al.* [12] have examined the much more difficult problem of flow past deformable drops in tubes. The major focus of all these studies has been the drag on the sphere (and deformations for [12]) rather than the stability of the computed axisymmetric flows.

In this paper we show how results from group theory applied to bifurcation problems can be used to reduce the complex matrix eigenvalue problem solved by Natarajan and Acrivos [6] to the real matrix eigenvalue problem investigated by Tavener [8]. We also show how an appropriately defined 'extended system' [13] may be used to calculate the symmetry breaking bifurcation points detected by both authors in a more robust and efficient manner. However, eigenvalue calculations are still required to ensure that the calculated bifurcation point corresponds to the instability that occurs for the lowest flow rate. It is important to note that even when there is a zero real eigenvalue with a corresponding singular Jacobian matrix, great care is needed when trying to locate this zero eigenvalue. The standard 'change in sign of the determinant' technique can be made to work only if an appropriately restricted Jacobian is used.

The importance of symmetry in bifurcation problems is well known and many of the key concepts, including the connection with group representation theory, were introduced or clarified in the books by Vanderbauwhede [14], Golubitsky and Schaeffer [15] and Golubitsky *et al.* [16]. The systematic numerical analysis of bifurcation in the presence of symmetry was started by Werner and Spence [13], Brezzi *et al.* [17] and Werner [18] for simple groups (e.g. Z_2). Dellnitz and Werner [19] and Werner [20] continued the detailed study for more general finite groups (e.g. D_n). Independently, Healey and his students [21–24], motivated by examples in structural mechanics, and Aston [25,26], motivated by a problem in water waves, developed similar theories but which allowed for infinite groups. An important idea is that, under the

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action of a group and for an appropriately chosen basis, the Jacobian of a non-linear problem 'block-diagonalizes', with consequent savings for numerical algorithms, and improved theoretical understanding [20,23,26]. Many of the group theoretic ideas can appear complicated, and one aim of this paper is to explain them as simply as possible for the $O(2)$ group, which is relevant for many problems defined on a cylindrical domain.

It is important to note that the results of this paper apply to all $O(2)$-symmetric problems and not just the example studied by Natarajan and Acrivos [6] and Tavener [8]. Group theoretic methods provide a rigorous justification of the considerable saving in computational expense when solving the linearized stability problem for all physical problems involving the breaking of an $O(2)$-symmetry. Similar savings in computational effort may be sought by the application of general group theoretic constructs to other more exotic dihedral or orthogonal symmetries (see, e.g. Dellnitz and Werner [19] and Dellnitz [27]). Here we simply illustrate their application to the familiar and common $O(2)$-symmetry group and demonstrate the advantages that can be accrued when studying the problem of flow past a sphere. More generally, once an underlying group theoretic structure is appreciated, very similar extended systems may be defined for the breaking of other symmetries. For example, Tavener *et al.* [28] efficiently compute the breaking of two distinct types of rotational invariances using this approach in their study of electro-hydrodynamic convection in nematic liquid crystals.

The detailed plan of the paper is as follows. In Section 2, we describe the general bifurcation theory for the particular example of $O(2)$-symmetry and explain in detail the technique for detecting both steady and Hopf bifurcations. In particular, we make the connection between the 'appropriately restricted Jacobian' and the 'block-diagonalization' in the previous paragraphs. In addition, we describe how a steady bifurcation may be accurately and efficiently computed using a generalization of the extended system presented in Werner and Spence [13], and how to decide if a bifurcating branch is sub- or supercritical. In Section 3 we apply these results to the Navier–Stokes equations and boundary conditions for the problem of flow past a sphere in a pipe, explaining in detail the equivariance condition and writing out explicitly the important components in the $O(2)$ theory. Section 4 contains a detailed discussion of the implementation of the extended system used to compute the bifurcation point in the physical example. In Section 5 we discuss the implications of the theory in Section 2 for the usual stability analysis, and hence give an improved stability analysis for the fluids problem that produces a real matrix eigenvalue problem directly. Numerical results are presented in Section 6, where the critical Reynolds number is plotted as a function of blockage ratio. A minor change in the boundary conditions allows the effect of a supporting wire, or sting, to be determined.

2. BIFURCATION THEORY IN THE PRESENCE OF $O(2)$-SYMMETRY

2.1. Preliminaries

Consider time-dependent non-linear problems of the form

$$By_t + f(y, \lambda) = 0, \quad y(t) \in H, \quad \lambda \in \mathbb{R}, \tag{1}$$

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where H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, f: $H \times \mathbb{R} \mapsto H$ is a smooth non-linear operator that is equivariant with respect to $O(2)$, B is a linear operator on H that is equivariant with respect to $O(2)$, and $y_t = \mathrm{d}y/\mathrm{d}t$. To be precise about the equivariance condition, recall that $O(2)$ is the Lie group generated by rotations r_α, $\alpha \in \mathbb{R}$ and a reflection s, satisfying, for any α, $\beta \in \mathbb{R}$,

$$r_{\alpha+2\pi} = r_\alpha, \qquad r_{\alpha+\beta} = r_\alpha r_\beta = r_\beta r_\alpha, \qquad s^2 = r_0 = r_{2\pi} = \mathbf{1}, \qquad s r_\alpha = r_{-\alpha} s, \tag{2}$$

where $\mathbf{1}$ is the group identity. An action of $O(2)$ on H is a continuous mapping ρ: $O(2) \times H \mapsto H$, $(\gamma, y) \mapsto \rho(\gamma, y) \equiv \gamma y$ such that $\mathbf{1}y = y$ and $(\gamma_1\gamma_2)y = \gamma_1(\gamma_2 y)$ for all $y \in H$, $\gamma_1, \gamma_2 \in O(2)$. For any action ρ of $O(2)$ on H we can define an orthogonal representation T in the space of linear homeomorphisms on H

$$T(\gamma)y \equiv \gamma y,$$

through the choice of an $O(2)$-invariant inner product

$$\langle T(\gamma)y_1, T(\gamma)y_2 \rangle = \langle y_1, y_2 \rangle \quad \forall y_1, y_2 \in H, \quad \forall \gamma \in O(2). \tag{3}$$

To say that B and f are equivariant with respect to the action of $O(2)$ means that

$$T(\gamma)B = BT(\gamma) \quad \forall \gamma \in O(2) \tag{4}$$

and

$$T(\gamma)f(y, \lambda) = f(T(\gamma)y, \lambda) \quad \forall \gamma \in O(2), \quad \forall y \in H, \tag{5}$$

where $T(\gamma)$ is a representation of $O(2)$. An immediate consequence of (4) and (5) is that if y solves (1) for a given λ, then so does $T(\gamma)y$ for all $\gamma \in O(2)$, giving a conjugacy class of solutions of (1).

A representation of $O(2)$ on a Hilbert space appropriate for the analysis of the Navier–Stokes equations in cylindrical co-ordinates is given in Section 3. A much simpler example is the linear equation

$$-u'' + \lambda u = 0, \tag{6}$$

with 2π periodic boundary conditions over $0 \leq x \leq 2\pi$. It is easy to see that this equation is equivariant with respect to the representation of $O(2)$ given by $T(r_\alpha)u(x) = u(x + \alpha)$ and $T(s)u(x) = -u(-x)$. This is the natural representation for (6) and is used by Aston [25] in the analysis and numerical solution of the Kuramoto–Sivashinsky equation [29]. Other examples involving the group $O(2)$ are found in Vanderbauwhede [14].

We are mainly interested in bifurcations from paths of steady $O(2)$symmetric solutions of (1). Denote the set of $O(2)$-symmetric elements of H by $H^{O(2)}$, i.e.

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$$H^{O(2)} := \{y \in H,\ y = T(\gamma)y,\ \forall \gamma \in O(2)\}.$$

An immediate consequence of (5) is that $H^{O(2)}$ is invariant under f since, for $y_{O(2)} \in H^{O(2)}$,

$$f(y_{O(2)}, \lambda) = f(T(\gamma)y_{O(2)}, \lambda) = T(\gamma)f(y_{O(2)}, \lambda), \tag{7}$$

and we may define $f_{O(2)}$ as the restriction of f to $H^{O(2)}$. Hence, steady $O(2)$-symmetric solutions of (1) and paths of such solutions, say

$$\mathscr{C} = \{(y(\lambda), \lambda):\ y(\lambda) \in H^{O(2)},\ \lambda \in I\},$$

where I is a given interval, may be computed by solving the reduced problem

$$f_{O(2)}(y_{O(2)}(\lambda), \lambda) = 0, \quad y_{O(2)} \in H^{O(2)}. \tag{8}$$

In Section 3 a problem defined on a cylindrical domain is considered, and in this case, the $O(2)$-symmetric solutions are axisymmetric, and have zero swirl.

Define $A(\lambda)$ to be the Jacobian matrix of f with respect to y along the path $\mathscr{C}$, i.e.

$$A(\lambda) := f_y(y(\lambda), \lambda), \quad (y(\lambda), \lambda) \in \mathscr{C}. \tag{9}$$

By differentiating (5) with respect to y we have

$$T(\gamma)f_y(y(\lambda), \lambda) = f_y(T(\gamma)y(\lambda), \lambda)T(\gamma)$$

and if $y(\lambda) \in H^{O(2)}$, then $T(\gamma)y(\lambda) = y(\lambda)$, producing

$$T(\gamma)A(\lambda) = A(\lambda)T(\gamma), \tag{10}$$

which is the same equivariance condition as that satisfied by B given by Equation (4).

It is a standard result (see, e.g. Aston [26], Theorems 2.6 and 2.11) that, for the special case of the group $O(2)$, there is a unique orthogonal decomposition of H

$$H = \sum_{m=0}^{\infty} \oplus V_m, \quad V_m \perp V_l, \quad m \neq l, \tag{11}$$

where V_m is an isotypic component of H, i.e. an $O(2)$-invariant subspace of H, with the additional property in this case that V_m is irreducible. In other words, for $y \in V_m$, we have $T(\gamma)y \in V_m$ for all $\gamma \in O(2)$ and V_m has no proper $O(2)$-invariant subspaces.

It is important to note that for a general group, the isotypic components are usually not irreducible, as can be seen in Aston [25], where an example involving the dihedral group D_4 is discussed in detail. However, the group $O(2)$, though infinite, has a rather simple structure and perhaps for this reason is not often studied in detail in bifurcation texts. An exception to this is the early book by Vanderbanwhede [14], which contains many interesting and informative examples with $O(2)$-symmetry.

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Equation (11) may be compared with the well-known decomposition of H under the action of Z_2, the group $\{1, s\}$, with s a reflection. Then H can be decomposed into symmetric and antisymmetric subspaces

$$H = H^s \oplus H^a, \tag{12}$$

where $y \in H^s \Rightarrow sy = y$, and $y \in H^a \Rightarrow sy = -y$.

For the 2π periodic example (6) mentioned in Section 2.1, the isotypic components are easily shown to be $V_0 = \text{span}\{1\}$, i.e. the constant functions, and

$$V_m = \text{span}\{\cos mx, \sin mx\}, \quad m = 1, 2, \ldots, \tag{13}$$

as one would expect from the theory of Fourier series. In fact, for the group $O(2)$, Equation (11) merely says that any element of H can be expressed uniquely as a Fourier series in $\cos mx$ and $\sin mx$. In this case, the invariance of V_m is easily shown, since if $u(x) = \cos mx$, then

$$T(r_\alpha) \cos mx = \cos m\alpha \cos mx - \sin m\alpha \sin mx, \tag{14}$$

and if $u(x) = \sin mx$ then

$$T(r_\alpha) \sin mx = \sin m\alpha \cos mx + \cos m\alpha \sin mx. \tag{15}$$

2.2. Eigenvalue theory for O(2)-equivariant linear operators

Standard linear stability arguments show that a linearly stable $O(2)$-symmetric solution will lose stability at $\lambda = \lambda_0$ when the generalized eigenvalue problem

$$\mu B\phi = A(\lambda)\phi, \tag{16}$$

where B and $A(\lambda)$ are linear operators on H, has an eigenvalue $\mu(\lambda_0)$ with $Re(\mu(\lambda_0)) = 0$ and $(d/d\lambda)[Re(\mu(\lambda_0))] \neq 0$. If $\mu(\lambda_0)$ is real then there is a steady bifurcation and if $\mu(\lambda_0)$ is complex then there is a Hopf bifurcation, but care is needed since the presence of $O(2)$-symmetry means that most of the eigenvalues of (16) occur with a multiplicity of two. However, we shall show that since B and $A(\lambda)$ are $O(2)$-equivariant, problem (16) may be reduced to one in which simple eigenvalues of a real generalized eigenvalue problem are sought.

Werner [20] provides a complete block diagonalization theory for the eigenvalue problem $A(\lambda)\phi = \mu\phi$, where $A(\lambda)$ is an $n \times n$ real matrix that satisfies an equivariance condition with respect to a compact Lie group of $n \times n$ orthogonal matrices. Aston [26] discusses equivariant bifurcation theory for general compact Lie groups, including $O(2)$, applied to problems defined on a Hilbert space. Healey and Treacy [23] present similar results for structures problems, and these were used by Wohlever and Healey [24] in an analysis of an axially compressed cylindrical shell. In this subsection we use the results of these three papers to give the theory for a generalized eigenvalue problem $\mu B\phi = A(\lambda)\phi$, where B and $A(\lambda)$ are $O(2)$-equivariant linear operators on a Hilbert space.

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The following theorem is a key result:

Theorem 2.2.1
Let A be any $O(2)$-equivariant linear operator on the Hilbert space H. Then A: $V_m \to V_m$, where the V_m are as in (11).

Proof
This result is a special case of Theorem 3.1 in Aston [26] (see also Section 2.3 of [20]). However, for the group $O(2)$ a straightforward proof by contradiction can be given. Assume that for some $y_m \in V_m$, $Ay_m = z_m + z_n$, with $z_m \in V_m$ and $z_n \in V_n$, $m \neq n$. Now, with $T(\gamma) = T(r_\alpha)$, for some $\alpha \in [0, 2\pi)$, and using (14) and (15), it is observed that $T(r_\alpha)y_m$ is a linear combination of $\cos m\alpha$ and $\sin m\alpha$. Thus, $AT(r_\alpha)y_m$ is also a linear combination of these terms. However, $T(r_\alpha)Ay_m = (\mathrm{T}(r_\alpha)z_m + T(r_\alpha)z_n)$ will be a linear combination of $\cos m\alpha$, $\sin m\alpha$, $\cos n\alpha$ and $\sin n\alpha$. This contradicts the identity $AT(r_\alpha) = T(r_\alpha)A$ $\forall\alpha$, unless $z_n = 0$. Clearly this argument extends to cover the assumption that $Ay_m = z_m + z_m^{\perp}$, $z_m^{\perp} \in V_m^{\perp}$. □

It is easily shown using (3) and (4) that if A is $O(2)$-equivariant then so is the adjoint operator A^*. Thus, we have the following corollary.

Corollary 2.2.1
Let A be any $O(2)$-equivariant linear operator on the Hilbert space H. Then A^*: $V_m \to V_m$, where A^* is the adjoint of A and the V_m are as in (11).

Since B and $A(\lambda)$ are both $O(2)$-equivariant, Theorem 2.2.1 shows that the eigenvalue problem $\mu B\phi = A(\lambda)\phi$, see (16), decouples into the (infinite) set of finite-dimensional generalized eigenvalue problems

$$\mu B_m\phi = A_m(\lambda)\phi, \quad \phi \in V_m, \quad m = 0, 1, 2, \ldots, \tag{17}$$

where B_m and $A_m(\lambda)$ are the restrictions of B and $A(\lambda)$ to V_m respectively. Equivalently, using the notation of [26],

$$\mu \,\mathrm{diag}(B_m)\phi = \mathrm{diag}(A_m(\lambda))\phi, \quad \phi \in H, \quad m = 0, 1, 2, \ldots, \tag{18}$$

and it is this decomposition that leads to the use of the term 'block-diagonalization', even when talking about infinite-dimensional operators (see, e.g. Healey and Treacy [23]).

Roughly speaking, one can think of the isotypic decomposition (11) as being produced by the 'rotation' elements r_α of $O(2)$. The fact that there is also a 'reflection' element has not yet been used, and indeed Theorem 3.3, Aston [26] applied to $O(2)$, shows there is a finer block decomposition

$$V_m = V_m^{\mathrm{s}} \oplus V_m^{\mathrm{a}}, \quad m = 1, 2, \ldots, \tag{19}$$

where V_m^{s} and V_m^{a} are the symmetric and anti-symmetric components of V_m, see (12). Further (Theorem 2.3 [20], Theorem 3.3 [26] and Figure 3 of [24]), V_m^{s} and V_m^{a} are invariant subspaces

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of B and $A(\lambda)$, and if we denote the restriction of $A_m(\lambda)$ to V^{s}_m (V^{a}_m) by A^{s}_m (A^{a}_m) and similarly for B_m, then for $m = 1, 2, \ldots$, Equation (18) can be further reduced to

$$\mu\begin{pmatrix} B^{\mathrm{s}}_m & 0 \\ 0 & B^{\mathrm{a}}_m \end{pmatrix}\begin{pmatrix} \phi^{\mathrm{s}} \\ \phi^{\mathrm{a}} \end{pmatrix} = \begin{pmatrix} A^{\mathrm{s}}_m & 0 \\ 0 & A^{\mathrm{a}}_m \end{pmatrix}\begin{pmatrix} \phi^{\mathrm{s}} \\ \phi^{\mathrm{a}} \end{pmatrix}, \quad m = 1, 2, \ldots, \tag{20}$$

with crucially

$$B^{\mathrm{s}}_m = B^{\mathrm{a}}_m \quad \text{and} \quad A^{\mathrm{s}}_m = A^{\mathrm{a}}_m, \quad m = 1, 2, \ldots . \tag{21}$$

Thus $A_m(\lambda)$, $m = 1, 2 \ldots$ is diagonalized into two identical sub-blocks and similarly for B_m. For the linear problem (6), the two identical sub-blocks are $0 = m^2 + \lambda$, $m = 1, 2 \ldots$

Returning to the eigenvalue problem $\mu B\phi = A(\lambda)\phi$, we have, therefore, proved the following lemma (see also Proposition 3.6 of [20]).

Lemma 2.2.1
Let $A(\lambda)$ and B be $O(2)$-equivariant.
(a) (μ, ϕ) is an eigenpair of $\mu B\phi = A(\lambda)\phi$ iff (μ, ϕ) is an eigenpair of $\mu B_m\phi = A_m(\lambda)\phi$, $\phi \in V_m$ for some m.
(b) If $m \neq 0$ in (a), then μ has a geometric multiplicity of 2, with the corresponding eigenvectors ϕ^{s} and ϕ^{a} satisfying $\mu B^{\mathrm{s}}_m\phi^{\mathrm{s}} = A^{\mathrm{s}}_m(\lambda)\phi^{\mathrm{s}}$ and $\mu B^{\mathrm{a}}_m\phi^{\mathrm{a}} = A^{\mathrm{a}}_m(\lambda)\phi^{\mathrm{a}}$, with $B^{\mathrm{s}}_m = B^{\mathrm{a}}_m$ and $A^{\mathrm{s}}_m(\lambda) = A^{\mathrm{a}}_m(\lambda)$.

Definition 2.2.1
Following Definition 3.7 in [20], we call μ an $O(2)$-simple eigenvalue if
(a) there is only one m for which μ is an eigenvalue of $\mu B_m\phi = A_m(\lambda)\phi$, and
(b) if in (a), $m = 0$, μ is an algebraically simple eigenvalue of $\mu B_0\phi = A_0(\lambda)\phi$, or if $m \neq 0$, μ is an algebraically simple eigenvalue of $\mu B^{\mathrm{s}}_m\phi = A^{\mathrm{s}}_m(\lambda)\phi$ (and $\mu B^{\mathrm{a}}_m\phi = A^{\mathrm{a}}_m(\lambda)\phi$).

It is now a simple matter (see Section 3.2 of [20]) to show that as the parameter λ, which describes the curve $\mathscr{C}$ varies, $O(2)$-simple eigenvalues of $\mu B\phi = A(\lambda)\phi$ behave like algebraically simple eigenvalues of matrices with no equivariance properties.

2.3. Steady state bifurcation theory

The bifurcation theory presented here is mainly the restriction of the results of Section 3 of [26] to the $O(2)$ case applied to (1).

Assume the curve of $O(2)$-symmetric solutions is being computed using (8). Further assume that $(y(\lambda_0), \lambda_0) \in \mathscr{C}$ and that $\mu(\lambda_0) = 0$ is an $O(2)$-simple eigenvalue of $\mu B\phi = \mathrm{A}(\lambda_0)\phi$. Let $\mathscr{N}_0 = \mathrm{Null}(A(\lambda_0))$. Clearly $\mathscr{N}_0 \subset V_m$ for some V_m from Lemma 2.2.1. If $\mathscr{N}_0 \in V_0$, i.e. $\mathscr{N}_0 \in H^{O(2)} \neq \{0\}$, then even if bifurcation occurs, the $O(2)$-symmetry will not be broken, hence this would be an $O(2)$-symmetry preserving bifurcation (generically, a turning point or fold point in the space of $O(2)$-symmetric functions). This case is of less interest to us and is excluded from the next theorem by the following assumption.

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We assume

Assumption 2.3.1
$\mathcal{N}_0 \subset V_m$, for some $m \neq 0$,

and

Assumption 2.3.2
μ is an $O(2)$-simple eigenvalue, i.e.

$$\dim(\mathcal{N}_0 \cap H^{\mathrm{s}}) = 1 \quad \text{and} \quad \dim(\mathcal{N}_0 \cap H^{\mathrm{a}}) = 1,$$

where $H^{\mathrm{s}} = \{y: sy = y\}$ and $H^{\mathrm{a}} = \{y: sy = -y\}$, the symmetric and anti-symmetric elements of H.

Before stating the main bifurcation result we first need to introduce some terms. We define the scalar b_λ by

$$b_\lambda := \langle \psi_0, f_{y\lambda}(y(\lambda_0), \lambda_0)\phi_0 + f_{yy}(y(\lambda_0), \lambda_0)\phi_0 v \rangle, \tag{22}$$

where $\phi_0 \in V_m^{\mathrm{s}} := V_m \cap H^{\mathrm{s}}$, $\psi_0 \in \mathrm{Null}(f_y(y(\lambda_0), \lambda_0)^*) \cap H^{\mathrm{s}}$, where * denotes the adjoint operator, and $v \in H^{O(2)}$ is the unique solution of

$$f_y(y(\lambda_0), \lambda_0)v + f_\lambda(y(\lambda_0), \lambda_0) = 0, \tag{23}$$

since $f_\lambda(y(\lambda_0), \lambda_0) \in H^{O(2)}$ for $y(\lambda_0) \in H^{O(2)}$. In fact if $v \in H^{O(2)}$ and $\phi_0 \in V_m^{\mathrm{s}}$, then $f_{y\lambda}(y(\lambda_0), \lambda_0), \lambda_0)\phi_0 \in V_m^{\mathrm{s}}$, and $f_{yy}(y(\lambda_0), \lambda_0)\phi_0 v \in V_m^{\mathrm{s}}$. (These are proved after differentiating Equation (5) appropriately.)

The next theorem provides the main bifurcation result. It is derived from Theorems 3.4 and 3.5 of [26] and Theorem 6.4.3 of [14].

Theorem 2.3.1
Let $y(\lambda_0) \in H^{O(2)}$. Suppose $A(\lambda_0)$ is the Fredholm of index zero and that Assumptions 2.3.1 and 2.3.2 hold. Assume the non-degeneracy condition

$$b_\lambda \neq 0. \tag{24}$$

Then there exists a secondary branch of $O(2)$-symmetry breaking solutions. The bifurcation is of pitchfork type and the bifurcating branch has D_m symmetry, where m is as in Assumption 2.3.1. (Here D_m is the dihedral group generated by $r_{2\pi/m}$ and s.)

Proof
The existence of a branch follows from the equivariant branching lemma [14,30] (see also [26]). The fact that the bifurcation is pitchfork follows from case (ii) of Theorem 3.5 of [26] since $\mathbf{N}_{O(2)}(Z_2)/Z_2$ is isomorphic to Z_2, where $\mathbf{N}_{O(2)}(Z_2)$ is the normalizer of Z_2 in $O(2)$ defined by

$$\mathbf{N}_{O(2)}(Z_2) = \{\gamma \in O(2): \ \gamma\sigma = \sigma\gamma \ \ \forall \sigma \in Z_2\}.$$

The fact that the bifurcating branch is in D_m is proved in Theorem 6.4.3 of [14]. □

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A test for steady bifurcations

Note that unless $m = 0$, a zero eigenvalue of $\mu B_m \phi = A_m \phi$ along a path of $O(2)$-symmetric solutions cannot be detected by looking for a change in sign in det $A(\lambda)$ or even det $A_m(\lambda)$, since two eigenvalues pass through zero at $\lambda = \lambda_0$. However, if only steady bifurcations are sought then a test based on detecting a sign change in det A_m^{s} (or in det A_m^{a}) would work since both these matrices have a simple eigenvalue at $\lambda = \lambda_0$.

Once a steady bifurcation has been located approximately, it may be calculated accurately using a generalization of the 'extended system' introduced by Werner and Spence [13] to compute pitchfork bifurcation points. The method is exactly as described in Theorem 3.6 of [26], which we specialize here to the $O(2)$ case.

Theorem 2.3.2

Let $(y(\lambda_0), \lambda_0) \in H^{O(2)} \times \mathbb{R}$ and assume Assumptions 2.3.1 and 2.3.2 hold with $\mathcal{N}_0 \subset V_m$ for some $m \neq 0$. Then, the extended system

$$G(z) = 0, \quad G\colon\ Z \mapsto Z, \tag{25}$$

where

$$G(z) = \begin{pmatrix} f(y, \lambda) \\ f_y(y, \lambda)\phi \\ \langle l, \phi \rangle - 1 \end{pmatrix}, \quad y \begin{pmatrix} y \\ \phi \\ \lambda \end{pmatrix} \in Z,$$

with $Z \in H^{O(2)} \times V_m^{\mathrm{s}} \times \mathbb{R}$, $l \in V_m^{\mathrm{s}}$, has an isolated solution (y_0, ϕ_0, λ_0), where $\phi_0 \in V_m^{\mathrm{s}}$ iff the non-degeneracy condition (24) holds.

Note that the theorem is true if V_m^{s} is replaced by V_m^{a} throughout. The system $G(z) = 0$ is known as an 'extended system'.

When answering questions on linearized stability of steady solutions of (1), it is often important to know if the bifurcating branch is subcritical or supercritical. To do this we need to calculate certain quantities involving higher derivatives of f_y. First note that by differentiating (5) and setting $y \in H^{O(2)}$ then $\forall T(\gamma)$,

$$T(\gamma) f_{yy}(y, \lambda)uv = f_{yy}(T(\gamma)y, \lambda)T(\gamma)uT(\gamma)v = f_{yy}(y, \lambda)T(\gamma)uT(\gamma)v.$$

Similarly, for $y \in H^{O(2)}$ and $\forall T(\gamma)$,

$$T(\gamma) f_{yyy}(y, \lambda)uvw = f_{yyy}(y, \lambda)T(\gamma)uT(\gamma)vT(\gamma)w.$$

Thus, with $\phi_0 \in V_m^{\mathrm{s}}$ (i.e. ϕ_0 is a multiple of $\cos mx$)

$$f_{yy}(y, \lambda)\phi_0\phi_0 \in H^{O(2)} \oplus V_{2m}^{\mathrm{s}},$$

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since $\cos^2 mx = (1 + \cos 2mx)/2$, and

$$f_{yyy}(y, \lambda)\phi_0\phi_0\phi_0 =: e_m + e_{3m} \in V^s_m \oplus V^s_{3m},$$

since $\cos^3 mx = (3\cos mx + \cos 3mx)/4$. We introduce $z = z_0 + z_{2m} \in H^{O(2)} \oplus V_{2m}$ respectively, where z is the unique solution of

$$f_y(y(\lambda_0), \lambda_0)z + f_{yy}(y(\lambda_0), \lambda_0)\phi_0\phi_0 = 0. \tag{26}$$

To decide on sub- or supercriticality we need to calculate b_λ in (22) and d, defined by (see Cliffe and Spence [31])

$$d := \langle \psi_0, f_{yyy}(y(\lambda_0), \lambda_0)\phi_0\phi_0\phi_0 + 3f_{yy}(y(\lambda_0), \lambda_0)\phi_0 z \rangle, \tag{27}$$

where ϕ_0 and ψ_0 are as in (22) and z is given by (26). Now, using Corollary 2.2.1, it is easily seen that ψ_0 is perpendicular to $V^\perp_m$. Thus, d can be found from

$$d := \langle \psi_0, e_m + 3f_{yy}(y(\lambda_0), \lambda_0)\phi_0 z_0 \rangle, \tag{28}$$

with e_m and z_0 defined above. Note that $d \neq 0$ ensures that there is a quadratic pitchfork bifurcation, whereas if $d = 0$ then there would be a quartic, or even more degenerate, pitchfork bifurcation (see Cliffe and Spence [31]). The following lemma provides the test for sub- or supercriticality.

Lemma 2.3.1
Assume $d/b_\lambda \neq 0$, where b_λ and d are given in (22) and (28) respectively. Then the bifurcation is supercritical if $d/b_\lambda < 0$, and subcritical if $d/b_\lambda > 0$.

Proof
In the proof of Theorem 2.3.1 it is shown that the pitchfork bifurcation is essentially of Z_2-symmetry breaking type. In Section 3 of [31] it is shown by a Lyapunov–Schmidt reduction procedure that the reduced equation for a Z_2-symmetry breaking bifurcation is

$$b_\lambda \lambda x + \frac{d}{6}x^3 + h.o.t = 0,$$

and so if $d/b_\lambda < 0$ this is (contact) equivalent to the normal form $\lambda x - x^3 = 0$, which has supercritical bifurcating branches. Similarly, $d/b_\lambda > 0$ leads to subcritical bifurcating branches. □

2.4. Time-dependent bifurcation theory

In this section we shall only discuss how to detect a Hopf bifurcation point on a path of $O(2)$-symmetric steady states of (1). For more details on the numerical analysis we refer the reader to [19,27], and to [16] for a more complete account of bifurcation theory for non-linear dynamical systems in the presence of $O(2)$-symmetry.

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We shall content ourselves here with noting that $O(2)$-symmetry breaking Hopf bifurcations can only occur at $\lambda = \lambda_0$ if $\mu B\phi = A(\lambda_0)\phi$ has a complex eigenvalue with $Re(\mu(\lambda_0)) = 0$. Lemma 2.2.1 applies here also and immediately we see that symmetry breaking Hopf bifurcation points at $\lambda = \lambda_0$ are only possible if, for $m \neq 0$, $\mu B^{\mathrm{s}}_m \phi^s = A^{\mathrm{s}}_m(\lambda_0)\phi^{\mathrm{s}}$ (or equivalently $\mu B^{\mathrm{a}}_m \phi^{\mathrm{a}} = A^{\mathrm{a}}_m(\lambda_0)\phi^{\mathrm{a}}$) has a purely imaginary eigenvalue pair. Assuming (see [20], Definition 4.12) that $\mu(\lambda_0)$ is simple and $(\mathrm{d}/\mathrm{d}\lambda)[Re(\mu(\lambda_0))] \neq 0$, then we say that (y_0, λ_0) is an $O(2)$-symmetry breaking Hopf bifurcation. Symmetry preserving Hopf bifurcations correspond to complex eigenvalues in the $m = 0$ block, $\mu B_0 \phi = A_0(\lambda_0)\phi$.

Since the bifurcation example of interest in Section 3 is a steady bifurcation, we do not consider this further here. We complete this section with a statement of a general means of detecting steady or Hopf bifurcations.

A test to detect a steady or Hopf bifurcations

If both steady and Hopf bifurcations are sought along a path of $O(2)$-symmetric steady solutions, then the eigenvalues nearest the imaginary axis need to be found for the problems $\mu B_0\phi = A_0\phi$, $\phi \in H^{O(2)}$, and $\mu B^{\mathrm{s}}_m \phi^{\mathrm{s}} = A^{\mathrm{s}}_m \phi^{\mathrm{s}}$, $m = 1, 2 \dots$. In practice one would choose an upper limit for m, say $m_{\max}$ and check the eigenvalues of the eigenvalue problems for $m = 0, 1, 2, \dots, m_{\max}$. For the numerical calculations in Section 5, $m_{\max}$ was taken to be 4.

An account of how this may be accomplished using a technique based on the Cayley transform [9] is given in Tavener [8].

3. THE FLUIDS PROBLEM

We now apply the abstract $O(2)$-symmetric bifurcation theory developed in Section 2 to the flow of a Newtonian fluid past a centrally located sphere in a pipe. In the usual cylindrical co-ordinates, the steady Navier–Stokes equations are

$$Re\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right) + \frac{\partial p}{\partial r} - \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) = 0, \quad (29)$$

$$Re\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right) + \frac{1}{r}\frac{\partial p}{\partial \theta} - \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) = 0, \quad (30)$$

$$Re\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) + \frac{\partial p}{\partial z} - \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right) = 0, \quad (31)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0. \quad (32)$$

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In order to non-dimensionalize the Navier–Stokes equations as above, the diameter of the pipe D was chosen as the length scale and the mean velocity

$$\bar{U} = \frac{4}{\pi D^2} \int_0^{D/2} 2\pi r u_z^* \, dr,$$

as the velocity scale, where u_z^* is the (dimensional) parabolic axial velocity that would exist in an unobstructed pipe. The Reynolds number $Re = D\bar{U}/v$, where v is the kinematic viscosity. We impose non-slip boundary conditions on the surface of the sphere and along the walls of the pipe, and assume that a fully developed parabolic velocity profile is attained sufficiently far upstream and downstream of the sphere.

Let $y(t)$ be a solution of Equations (29)–(32), i.e.

$$y(t) = \begin{pmatrix} u_r(r, \theta, z) \\ u_\theta(r, \theta, z) \\ u_z(r, \theta, z) \\ p(r, \theta, z) \end{pmatrix},$$

where $y(t)$ lies in the Hilbert space $H = W^{1,2}(\Omega)^3 \times L^2(D)$. Equations (29)–(32) may be combined to define a mapping

$$f(y, Re)\colon\ H \times \mathbb{R} \to H. \tag{33}$$

The bounded linear operators

$$R_\alpha \begin{pmatrix} u_r(r, \theta, z) \\ u_\theta(r, \theta, z) \\ u_z(r, \theta, z) \\ p(r, \theta, z) \end{pmatrix} = \begin{pmatrix} u_r(r, \theta+\alpha, z) \\ u_\theta(r, \theta+\alpha, z) \\ u_z(r, \theta+\alpha, z) \\ p(r, \theta+\alpha, z) \end{pmatrix}, \quad \alpha \in [0, 2\pi), \tag{34}$$

and

$$S \begin{pmatrix} u_r(r, \theta, z) \\ u_\theta(r, \theta, z) \\ u_z(r, \theta, z) \\ p(r, \theta, z) \end{pmatrix} = \begin{pmatrix} u_r(r, -\theta, z) \\ -u_\theta(r, -\theta, z) \\ u_z(r, -\theta, z) \\ p(r, -\theta, z) \end{pmatrix} \tag{35}$$

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are representations of the symmetry group $O(2)$ on the Hilbert space H. It is straightforward to show that the operator f defined by Equation (33) is equivariant with respect to both R_α and S, i.e.

$$R_\alpha f(y, Re) = f(R_\alpha y, Re) \tag{36}$$

and

$$Sf(y, Re) = f(Sy, Re). \tag{37}$$

Applying the theory in Section 2.2, especially Equations (11) and (13) (see also Aston [25]), the solution space H may be decomposed as an orthogonal direct sum of finite-dimensional $O(2)$-irreducible subspaces V_m,

$$H = V_0 \oplus V_1 \oplus V_2 \oplus \cdots,$$

where

$$V_m = \text{span}\left\{ \begin{pmatrix} u_r^m(r, z)\cos m\theta \\ u_\theta^m(r, z)\sin m\theta \\ u_z^m(r, z)\cos m\theta \\ p^m(r, z)\cos m\theta \end{pmatrix}, \begin{pmatrix} u_r^m(r, z)\sin m\theta \\ u_\theta^m(r, z)\cos m\theta \\ u_z^m(r, z)\sin m\theta \\ p^m(r, z)\sin m\theta \end{pmatrix} \right\}. \tag{38}$$

A (matrix) representation of $O(2)$ on the irreducible subspace V_m is

$$R_\alpha^m = \begin{pmatrix} C & D \\ -D & C \end{pmatrix},$$

where

$$C = \begin{bmatrix} \cos m\alpha & 0 & 0 & 0 \\ 0 & \cos m\alpha & 0 & 0 \\ 0 & 0 & \cos m\alpha & 0 \\ 0 & 0 & 0 & \cos m\alpha \end{bmatrix},$$

$$D = \begin{bmatrix} \sin m\alpha & 0 & 0 & 0 \\ 0 & -\sin m\alpha & 0 & 0 \\ 0 & 0 & \sin m\alpha & 0 \\ 0 & 0 & 0 & \sin m\alpha \end{bmatrix},$$

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and

$$S^m = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix},$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notice that each isotypic subspace is two($\times$ 4)-dimensional.

Due to the symmetrical nature of the physical domain and to the boundary conditions imposed, there are solutions of the flow problem that are $O(2)$-invariant. Let $H^{O(2)}$ be the subspace of $O(2)$-invariant solutions. Following the theory in Section 2 we define

$$f^{O(2)}\colon\ H^{O(2)} \to H^{O(2)} \tag{39}$$

to be the restriction of f to $H^{O(2)}$.

In order to compute $O(2)$-invariant solutions we solved Equations (29)–(32) in the domain Ω, shown in Figure 1, which is a radial slice through the physical domain. Non-slip velocity boundary conditions were imposed on the surface of the sphere and along the walls of the pipe. At the inflow boundary only the axial component of the velocity field is non-zero and $u_z(r)$ is both axisymmetric and time independent. The Dirichlet boundary conditions applied were, therefore,

$$u_r = u_\theta = u_z = 0 \ \text{ on } \ \partial\Omega_{\text{pipe}},$$

$$u_r = u_\theta = u_z = 0 \ \text{ on } \ \partial\Omega_{\text{sphere}},$$

$$u_r = u_\theta = 0, \quad u(z) = 2(0.5 - r)(0.5 + r) \ \text{ on } \ \partial\Omega_{\text{in}}. \tag{40}$$

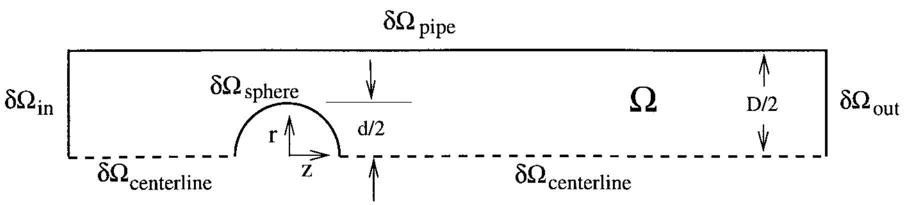

Figure 1. Sketch of the physical domain.

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The boundary conditions along the remaining boundaries of Ω were chosen to be those arising as natural boundary conditions when a weak formulation of Equations (29)–(32) is constructed, namely

$$-p+\frac{\partial u_z}{\partial z}=0,\quad \frac{\partial u_r}{\partial z}=\frac{\partial u_\theta}{\partial z}=0 \text{ on } \partial\Omega_{\text{out}},$$

$$-p+\frac{\partial u_r}{\partial r}=0,\quad \frac{\partial u_\theta}{\partial r}=\frac{\partial u_z}{\partial r}=0 \text{ on } \partial\Omega_{\text{centerline}}. \tag{41}$$

As a consequence of Equation (36) and (37), for $y \in H^{O(2)}$, we can use Equations (17) and (18) to decompose the Fréchet derivative $f_y(y, Re)$ into block-diagonal form

$$f_y(y, Re)=\text{diag}(f_y^m(y, Re)),\quad m=0, 1, 2, \ldots,$$

where $f_y^m=f_y\big|_{Vm}$ and $f_y^m\colon V_m\to V_m$.

Using Equations (19)–(21) we can further decompose the block-diagonal structure of f_y for $m\neq 0$. To avoid a double subscript notation we define $g_y^m\colon V_m^{\text{s}}\mapsto V_m^{\text{s}}$ and $h_y^m\colon V_m^{\text{a}}\mapsto V_m^{\text{a}}$, such that

$$f_y^m=\text{diag}(g_y^m, h_y^m),\quad m\neq 0,$$

where g_y^m and h_y^m are identical.

In this way we reduce the problem of finding a singular point of f_y to that of finding a singular point of g_y^m for some m. If g_y^m is singular for any $m\neq 0$, then Theorem 2.3.1 states that there will be a bifurcating branch of D_m-invariant solutions, i.e. solutions that are invariant with respect to S and $R_{2\pi/m}$, and that the symmetry breaking will occur at a pitchfork bifurcation point, which may be calculated using the extended system (25).

4. SOLUTION PROCEDURE

We construct the weak form of Equations (29)–(32) and boundary conditions (40) and (41) and discretize the $O(2)$-symmetric subspace $H^{O(2)}$ using a finite element approximation in a radial slice through the full cylindrical domain. The finite-dimensional approximating subspace $\boldsymbol{H}^{O(2)}\subset H^{O(2)}$ of dimension N, say, was constructed using quadrilateral elements with biquadratic velocity interpolation and discontinuous linear pressure interpolation using the finite element code ENTWIFE [32]. Let $\boldsymbol{f}\colon \boldsymbol{H}^{O(2)}\to\boldsymbol{H}^{O(2)}$ be the discretized weak form of the restriction of f to $H^{O(2)}$. Further, let $\boldsymbol{V}_m^{\text{s}}$ and $\boldsymbol{V}_m^{\text{a}}$ be the discretizations versions of subspaces V_m^{s} and V_m^{a} respectively and $\boldsymbol{g}_y^m$ and $\boldsymbol{h}_y^m$ be the discretized versions of Fréchet derivatives g_y^m and h_y^m respectively.

We locate *steady* symmetry breaking bifurcation points by constructing a discretized version of the extended system (25) as follows.

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Find $\boldsymbol{y}\in\boldsymbol{H}^{O(2)}$, $\vec{\phi}\in\boldsymbol{V}_m^{\mathrm{s}}$ and $\lambda\in R$ such that

$$\boldsymbol{G}(\boldsymbol{z}) = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g}_{\boldsymbol{y}}^{m}\vec{\phi} \\ \boldsymbol{l}^{T}\vec{\phi} - 1 \end{pmatrix} = \boldsymbol{0}, \tag{42}$$

where $\boldsymbol{z} = (\boldsymbol{y}, \vec{\phi}, \lambda) = (\boldsymbol{H}^{O(2)} \times \boldsymbol{V}_m^{\mathrm{s}} \times \mathbb{R})\in\boldsymbol{Z}$ and $\boldsymbol{G}$: $\boldsymbol{Z}\mapsto\boldsymbol{Z}$. Note that $\dim(\boldsymbol{Z}) = 2N + 1$.

We could equally well have chosen to solve the problem: find $\boldsymbol{y}\in\boldsymbol{H}^{O(2)}$, $\vec{\phi}\in\boldsymbol{V}_m^{\mathrm{a}}$ and $\lambda\in\mathbb{R}$ such that

$$\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{h}_{\boldsymbol{y}}^{m}\vec{\phi} \\ \boldsymbol{l}^{T}\vec{\phi} - 1 \end{pmatrix} = \boldsymbol{0}.$$

The extended system (42) is a system of size $(2N + 1)$ and it might seem that when solving (42) by Newton's method one would have to actually solve a sequence of $(2N + 1) \times (2N + 1)$ linear systems. In fact as was shown in Werner and Spence [13], these systems can be solved efficiently by a procedure involving the solution of a system of size N with two right-hand sides, and a system of size $(N + 1)$ with a single right-hand side.

In many cases one wishes to calculate a *path* of symmetry breaking bifurcations. We now describe an appropriate solution procedure to compute such a path based on the technique in Werner and Spence [13], which only requires additional back-substitutions. Consider the two parameter problem $f(y, \lambda, \sigma) = 0$. Assume that for fixed σ a bifurcation point has been calculated using (42). Now, as σ varies a path of symmetry breaking bifurcations will be described, given by the solutions of the following system, where $\boldsymbol{y}\in\boldsymbol{H}^{O(2)}$, $\vec{\phi}\in\boldsymbol{V}_m^{\mathrm{s}}$ and $\lambda\in\mathbb{R}$:

$$\boldsymbol{G}(\boldsymbol{z}, \sigma) = \begin{pmatrix} \boldsymbol{f}(\boldsymbol{y}, \lambda, \sigma) \\ \boldsymbol{g}_{\boldsymbol{y}}^{m}(\boldsymbol{y}, \lambda, \sigma)\vec{\phi} \\ \boldsymbol{l}^{T}\vec{\phi} - 1 \end{pmatrix} = \boldsymbol{0}, \tag{43}$$

where $\boldsymbol{z} = (\boldsymbol{y}, \vec{\phi}, \lambda) = (\boldsymbol{H}^{O(2)} \times \boldsymbol{V}_m^{\mathrm{s}} \times \mathbb{R})\in\boldsymbol{Z}$ and $\boldsymbol{G}$: $\boldsymbol{Z} \times \mathbb{R}\mapsto\boldsymbol{Z}$.

The path following could be accomplished as follows. For an initial guess $(\boldsymbol{z}_0, \sigma_0)$ such that $\boldsymbol{G}(\boldsymbol{z}_0, \sigma_0) \neq \boldsymbol{0}$, we seek $\Delta\boldsymbol{z}$ and $\Delta\sigma$ such that $\boldsymbol{G}(\boldsymbol{z}_0 + \Delta\boldsymbol{z}, \sigma_0 + \Delta\sigma) = \boldsymbol{0}$. Linearizing about $(\boldsymbol{z}_0, \sigma_0)$.

$$\boldsymbol{G}(\boldsymbol{z}_0 + \Delta\boldsymbol{z}, \sigma_0 + \Delta\sigma) \approx \boldsymbol{G}(\boldsymbol{z}_0, \sigma_0) + \boldsymbol{G}_{\boldsymbol{z}}(\boldsymbol{z}_0, \sigma_0)\Delta\boldsymbol{z} + \boldsymbol{G}_{\sigma}(\boldsymbol{z}_0, \sigma_0)\Delta\sigma = \boldsymbol{0}$$

or

$$\boldsymbol{G}_{\boldsymbol{z}}(\boldsymbol{z}_0, \sigma_0)\Delta\boldsymbol{z} + \boldsymbol{G}_{\sigma}(\boldsymbol{z}_0, \sigma_0)\Delta\sigma = -\boldsymbol{G}(\boldsymbol{z}_0, \sigma_0).$$

Applied to (43) this is

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$$\begin{pmatrix} \boldsymbol{f}_y & 0 & \boldsymbol{f}_\lambda \\ \boldsymbol{g}^m_{yy}\vec{\phi} & \boldsymbol{g}^m_y & \boldsymbol{g}^m_{y\lambda}\vec{\phi} \\ 0 & \boldsymbol{l}^T & 0 \end{pmatrix} \begin{pmatrix} \Delta\boldsymbol{y} \\ \Delta\vec{\phi} \\ \Delta\lambda \end{pmatrix} + \begin{pmatrix} \boldsymbol{f}_\sigma \\ \boldsymbol{g}^m_{y\sigma}\vec{\phi} \\ 0 \end{pmatrix} \Delta\sigma = - \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g}^m_y\vec{\phi} \\ \boldsymbol{l}^T\vec{\phi}-1 \end{pmatrix},$$

where the derivatives are evaluated at (z_0, σ_0). Note that $\boldsymbol{f}_y = \boldsymbol{g}^0_y$, the Jacobian restricted to the $O(2)$-symmetric space $\boldsymbol{H}^{O(2)}$.

At a symmetry breaking bifurcation point $m \neq 0$, so the matrix $\boldsymbol{f}_y$ is invertible on $\boldsymbol{H}^{O(2)}$ and $\boldsymbol{g}^m_y$ is singular on $\boldsymbol{V}^s_m$. Letting

$$\Delta\boldsymbol{y} = \boldsymbol{a} + \boldsymbol{b}\Delta\lambda + \boldsymbol{c}\Delta\sigma, \tag{44}$$

the equation

$$\boldsymbol{f}_y\Delta\boldsymbol{y} + \boldsymbol{f}_\lambda\Delta\lambda + \boldsymbol{f}_\sigma\Delta\sigma = -\boldsymbol{f} \tag{45}$$

becomes

$$\boldsymbol{f}_y(\boldsymbol{a} + \boldsymbol{b}\Delta\lambda + \boldsymbol{c}\Delta\sigma) + \boldsymbol{f}_\lambda\Delta\lambda + \boldsymbol{f}_\sigma\Delta\sigma = -\boldsymbol{f}$$

or

$$\boldsymbol{f}_y\boldsymbol{a} + (\boldsymbol{f}_y\boldsymbol{b} + \boldsymbol{f}_\lambda)\Delta\lambda + (\boldsymbol{f}_y\boldsymbol{c} + \boldsymbol{f}_\sigma)\Delta\sigma = -\boldsymbol{f},$$

and we can solve

$$\boldsymbol{f}_y\boldsymbol{a} = -\boldsymbol{f}, \tag{46}$$

$$\boldsymbol{f}_y\boldsymbol{b} = -\boldsymbol{f}_\lambda, \tag{47}$$

$$\boldsymbol{f}_y\boldsymbol{c} = -\boldsymbol{f}_\sigma, \tag{48}$$

for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{H}^{O(2)}$. Thus, we factorize a single $N \times N$ dimensional matrix and solve for three different right-hand sides.

Since $\boldsymbol{g}^m_y$ is not invertible on V_m, the second and third rows in the extended system

$$\boldsymbol{g}^m_{yy}\vec{\phi}\Delta\boldsymbol{y} + \boldsymbol{g}^m_y\Delta\vec{\phi} + \boldsymbol{g}^m_{y\lambda}\Delta\lambda + \boldsymbol{g}^m_{y\sigma}\Delta\sigma = -\boldsymbol{g}^m_y\vec{\phi}, \tag{49}$$

$$\boldsymbol{l}^T\Delta\vec{\phi} = \boldsymbol{l}^T\vec{\phi} - 1 \tag{50}$$

are solved together. Letting

$$\Delta\vec{\phi} = \boldsymbol{d} + \boldsymbol{e}\Delta\sigma, \tag{51}$$

$$\Delta\lambda = \alpha + \beta\Delta\sigma, \tag{52}$$

we have

$$\Delta y = \boldsymbol{a} + \boldsymbol{b}\Delta\lambda + \boldsymbol{c}\Delta\sigma = \boldsymbol{a} + \boldsymbol{b}(\alpha + \beta\Delta\sigma) + \boldsymbol{c}\Delta\sigma = (\boldsymbol{a} + \alpha\boldsymbol{b}) + (\beta\boldsymbol{b} + \boldsymbol{c})\Delta\sigma. \tag{53}$$

Substituting Equations (51)–(53) into Equation (49), we have

$$\boldsymbol{g}_{yy}^{m}\vec{\phi}[(\boldsymbol{a} + \alpha\boldsymbol{b}) + (\beta\boldsymbol{b} + \boldsymbol{c})\Delta\sigma] + \boldsymbol{g}_{y}^{m}(\boldsymbol{d} + \boldsymbol{e}\Delta\sigma) + \boldsymbol{g}_{y\lambda}^{m}\vec{\phi}(\alpha + \beta\Delta\sigma) + \boldsymbol{g}_{y\sigma}^{m}\vec{\phi}\Delta\sigma = -\boldsymbol{g}_{y}^{m}\vec{\phi}.$$

Hence

$$\boldsymbol{g}_{yy}^{m}\vec{\phi}\boldsymbol{a} + \alpha\boldsymbol{g}_{yy}^{m}\vec{\phi}\boldsymbol{b} + \boldsymbol{g}_{y}^{m}\boldsymbol{d} + \alpha\boldsymbol{g}_{y\lambda}^{m}\vec{\phi} + [\beta\boldsymbol{g}_{yy}^{m}\vec{\phi}\boldsymbol{b} + \boldsymbol{g}_{yy}^{m}\vec{\phi}\boldsymbol{c} + \boldsymbol{g}_{y}^{m}\boldsymbol{e} + \beta\boldsymbol{g}_{y\lambda}^{m}\vec{\phi} + \boldsymbol{g}_{y\sigma}^{m}\vec{\phi}]\Delta\sigma = -\boldsymbol{g}_{y}^{m}\vec{\phi},$$

or

$$\boldsymbol{g}_{y}^{m}\boldsymbol{d} + \alpha(\boldsymbol{g}_{yy}^{m}\vec{\phi}\boldsymbol{b} + \boldsymbol{g}_{y\lambda}^{m}\vec{\phi}) = -\boldsymbol{g}_{y}^{m}\vec{\phi} - \boldsymbol{g}_{yy}^{m}\vec{\phi}\boldsymbol{a} \tag{54}$$

and

$$\boldsymbol{g}_{y}^{m}\boldsymbol{e} + \beta(\boldsymbol{g}_{yy}^{m}\vec{\phi}\boldsymbol{b} + \boldsymbol{g}_{y\lambda}^{m}\vec{\phi}) = -\boldsymbol{g}_{y\sigma}^{m}\vec{\phi} - \boldsymbol{g}_{yy}^{m}\vec{\phi}\boldsymbol{c}. \tag{55}$$

Substituting (51) into (50) we have

$$\boldsymbol{l}^{T}(\boldsymbol{d} + \boldsymbol{e}\Delta\sigma) = \boldsymbol{l}^{T}\vec{\phi} - 1$$

or

$$\boldsymbol{l}^{T}\boldsymbol{d} = \boldsymbol{l}^{T}\vec{\phi} - 1 \tag{56}$$

and

$$\boldsymbol{l}^{T}\boldsymbol{e} = 0. \tag{57}$$

Given $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as solutions to (46), (47) and (48) respectively, we solve (54) and (56) together for $\boldsymbol{d}$ and α, and (55) and (57) together for $\boldsymbol{e}$ and β. Note that we solve a single $(N+1)\times(N+1)$ dimensional system with two different right-hand sides. We then compute $\Delta\vec{\phi}$ and $\Delta\lambda$ and finally $\Delta\boldsymbol{y}$ by substituting into Equations (51)–(53) respectively.

To compute d/b_{λ} as required in Lemma 2.3.1, we first solve $\boldsymbol{f}_{y}\boldsymbol{z} = \boldsymbol{g}_{yy}^{m}\vec{\phi}\vec{\phi}$ for $\boldsymbol{z}\in\boldsymbol{H}^{O(2)}$, then calculate the ratio

$$\frac{\psi^{T}(3\boldsymbol{g}_{yy}^{m}\vec{\phi}\boldsymbol{z} + 3\boldsymbol{g}_{yyy}^{m}\vec{\phi}\vec{\phi}\vec{\phi})}{\psi^{T}(3\boldsymbol{g}_{yy}^{m}\vec{\phi}\boldsymbol{b} + 3\boldsymbol{g}_{y\lambda}^{m}\vec{\phi})},$$

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where $\boldsymbol{f}_y\boldsymbol{b} = -\boldsymbol{f}_\lambda$, see Equation (47). It is worth mentioning that a change in the normalization of ϕ will change the value of d/b_λ, but does not change its sign, since if $\phi \to \beta\phi$ for some real β, then $z \to \beta^2 z$ and $d/b_\lambda \to \beta^2 d/b_\lambda$.

Starting values for the Newton iteration were determined from the previous linear stability calculations reported by Tavener [8].

5. COMPARISON OF THEORY IN SECTION 2 TO THE STANDARD APPROACH

Natarajan and Acrivos [6], Tavener [8] and Kim and Pearlstein [33] carried out a stability analysis for the problem in Section 3. Let

$$(u_r, u_\theta, u_z, p) = (u_r^0(r, z), 0, u_z^0(r, z), p^0(r, z)) \in H^{O(2)}$$

denote a steady $O(2)$-symmetric solution of Equations (29)–(32). These authors considered perturbations of the form

$$u_r = u_r^0(r, z) + \epsilon u_r^1(r, z)\, \mathrm{e}^{im\theta - \mu t}, \ldots \text{ etc.},$$

and derived a *complex* matrix eigenvalue problem for each m. In all three papers it was assumed, or stated without proof, that the perturbations do not introduce any mixed mode terms in the linearization. Since the linearization satisfies the equivariance condition (10), Theorem 2.2.1 applies and shows that this assumption is indeed correct. In addition, for $m \neq 0$, Tavener [8] showed how the complex matrix eigenvalue problem may be transformed to a real eigenvalue problem by simple manipulation, namely by multiplying the 'u_θ^1' equation by i. Lemma 2.2.1(b) shows that it will always be possible to make a transformation to a real matrix eigenvalue problem for any $O(2)$-symmetric problem.

In fact, we now give a stability analysis that produces the real matrix eigenvalue problem directly using our knowledge of the decomposition of H induced by the action of $O(2)$ given by Equation (38).

Consider a nearby solution of Equations (29)–(32) of the form

$$\begin{aligned}(u_r^0, 0, u_z^0, p^0) + \epsilon[&u_r^1(r, z)\,\mathrm{e}^{-\mu t}(\cos m\theta + i\sin m\theta),\ u_\theta^1(r, z)\,\mathrm{e}^{-\mu t}(\sin m\theta + i\cos m\theta)\\ \times,\ &u_z^1(r, z)\,\mathrm{e}^{-\mu t}(\cos m\theta + i\sin m\theta),\ p^1(r, z)e^{-\mu t}(\cos m\theta + i\sin m\theta)],\end{aligned} \tag{58}$$

where $0 < \epsilon \ll 1$, m is a positive integer and μ is complex. Substituting the new solution into the equilibrium equations (29)–(32) and equating to zero terms that are real and first-order in ϵ, the linearized equations are

$$\begin{aligned}&R\left(-\mu u_r^1 + u_r^0\frac{\partial u_r^1}{\partial r} + u_r^1\frac{\partial u_r^0}{\partial r} + u_z^0\frac{\partial u_r^1}{\partial z} + u_r^1\frac{\partial u_r^0}{\partial r}\right) + \frac{\partial p^1}{\partial r}\\ &-\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r^1}{\partial r}\right) - \frac{m^2}{r^2}u_r^1 + \frac{\partial^2 u_r^1}{\partial z^2} - \frac{\partial u_r^1}{r^2} - \frac{2m}{r^2}u_\theta^1\right) = 0,\end{aligned} \tag{59}$$

$$R\left(-\mu u_\theta^1 + u_r^0 \frac{\partial u_\theta^1}{\partial r} + u_z^0 \frac{\partial u_\theta^1}{\partial z} + \frac{u_r^0 u_\theta^1}{r}\right) - \frac{m}{r} p^1$$

$$-\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_\theta^1}{\partial r}\right) - \frac{m^2}{r^2} u_\theta^1 + \frac{\partial^2 u_\theta^1}{\partial z^2} + \frac{2m}{r^2} u_r^1 - \frac{u_\theta^1}{r^2}\right) = 0, \tag{60}$$

$$R\left(-\mu u_z^1 + u_r^0 \frac{\partial u_z^1}{\partial r} + u_r^1 \frac{\partial u_z^0}{\partial r} + u_z^0 \frac{\partial u_z^1}{\partial z} + u_z^1 \frac{\partial u_z^0}{\partial z}\right) + \frac{\partial p^1}{\partial z} - \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z^1}{\partial r}\right) - \frac{m^2}{r^2} u_z^1 + \frac{\partial^2 u_z^1}{\partial z^2}\right)$$

$$= 0, \tag{61}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r^1) + \frac{m}{r} u_\theta^1 + \frac{\partial u_z^1}{\partial z} = 0, \tag{62}$$

which involve no imaginary terms. An identical set of equations is obtained by equating the terms that are imaginary and first-order in ϵ to be zero.

6. RESULTS

The critical Reynolds number *Re* obtained at a regular solution of the extended system (42) for the flow past a centrally located sphere in a pipe is given as a function of blockage ratio (BR) in Figure 2. For the purposes of comparison with experiments and computations of the

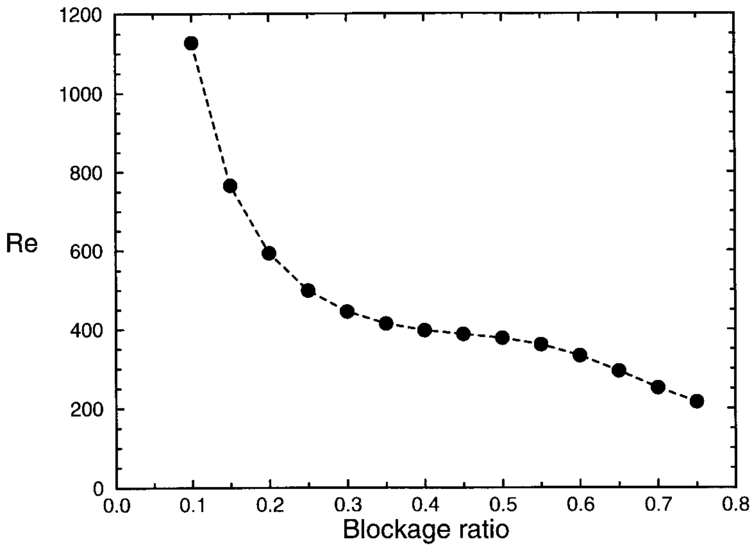

Figure 2. Critical Reynolds number, *Re* vs. BR for the flow past a sphere in a pipe.

exterior flow past spheres in 'unbounded' domains, we define a second (dependent) Reynolds number Re_d based on the sphere diameter, and the axial inlet velocity $u_z^*(r)$ averaged across the sphere cross-section. Let $Re_d = d\bar{U}/\nu$, where

$$\bar{U}_d = \frac{4}{\pi d^2}\int_0^{d/2} 2\pi r u_z^*(r)\,dr,$$

with $u_z^*(r)$ as in Section 3. The corresponding critical Reynolds number Re_d is given as a function of blockage ratio in Table I and Figure 3. Our results are consistent with those of Tavener [8], Figure 5. A finite element mesh with a length of three pipe diameters upstream of the sphere and eight pipe diameters downstream of the sphere incorporating 4400 quadrilateral elements was used to obtain these values. Based on the studies shown in Tables II and III, we believe the critical Reynolds number to be accurate to within 1%.

The result of two consecutive mesh halvings is given in Table II for a blockage ratio of 0.5. It is evident from Table III, that the effect of domain length downstream of the sphere is weak and that even for very short domains, the estimate of the critical Reynolds number is surprisingly good. This, and the null eigenvectors themselves (as presented previously by both Natarajan and Acrivos [6] and Tavener [8]) indicate that the instability is localized in the sphere wake.

As predicted by the theory in Section 2, the determinant of the Jacobian $\boldsymbol{f}_y$ did *not* change sign along the $O(2)$-symmetric solution branches at the symmetry breaking bifurcation points. Previous eigenvalue computations by [8] were necessary to provide an initial guess for the extended system (42). An assumption of smoothness with respect to the blockage ratio was used to provide an initial guess for the critical Reynolds number for blockage ratios at which eigenvalue computation had not been performed.

The sign of the ratio d/b_λ (Table II and indeed for all computations) indicates that these pitchfork bifurcations are supercritical (see Lemma 2.3.1), a result that is consistent with both the experimental evidence of Margarvey and Bishop [3], Nakamura [4] and Wu and Faeth [5], the computational evidence of Tomboulides *et al.* [7] and the conjecture of Natarajan and Acrivos [6]. It could not be determined by the earlier eigenvalue computations alone.

Table I. Critical Re_d vs. blockage ratio (BR)

BR	Critical Re_d	BR	Critical Re_d
0.10	224.4	0.45	313.4
0.15	227.2	0.50	330.3
0.20	232.6	0.55	337.0
0.25	241.5	0.60	327.7
0.30	254.6	0.65	301.6
0.35	271.9	0.70	266.3
0.40	292.4	0.75	232.8

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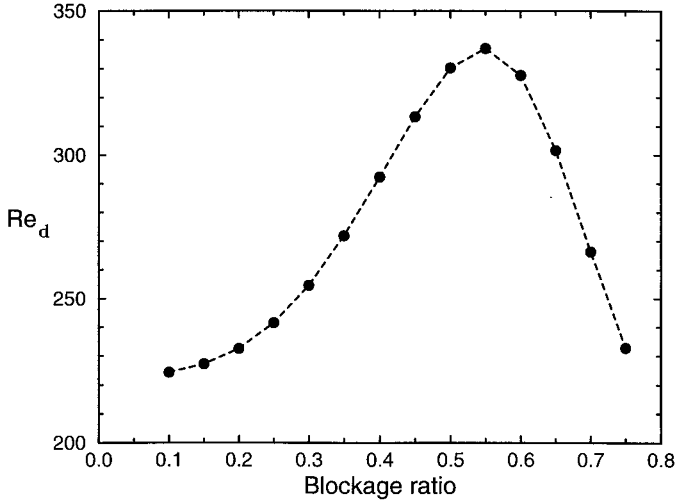

Figure 3. Critical Reynolds number, Re_d vs. BR for the flow past a sphere in a pipe.

Table II. Convergence study for BR 0.5

Number of elements	Critical Re_d	d/b_λ
275	343.4	−34.83
1100	333.0	−31.87
4400	330.3	−33.39

Table III. Effect of domain length for BR 0.75 using a mesh with 1331 elements

Domain length downstream of sphere (pipe diameters)	Critical Re_d
3.5	233.6
5.5	233.7
8	233.8
10.5	234.0
13	234.2

Wham *et al.* [11] show that the presence of a rear-facing sting or wire that is sometimes used in experiments to support a sphere makes only a 3% difference to the drag on the sphere. Encouraged by this result, we repeated our computational procedure setting $u = v = 0$ along

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the centerline of the pipe downstream of the sphere to model the presence of a wire. As can been seen from Table IV, we observed a change in the critical Reynolds number at the $O(2)$-symmetry breaking bifurcation point of between 3% and 5%. The bifurcation remains supercritical in nature for all blockage ratios examined.

Plots of the critical Reynolds numbers Re and Re_d as a function of blockage ratio are shown in Figures 4 and 5 respectively. These results were obtained on the same grid as that used for the unsupported sphere. Based on a second convergence study (Table V) we again believe that the critical Reynolds numbers are accurate to within 1%.

Table IV. Critical Re_d vs. BR

BR	Critical Re_d	BR	Critical Re_d
0.10	235.9	0.45	322.7
0.15	238.2	0.50	339.9
0.20	243.0	0.55	346.8
0.25	251.3	0.60	337.5
0.30	263.9	0.65	310.4
0.35	281.0	0.70	273.1
0.40	301.5	0.75	237.6

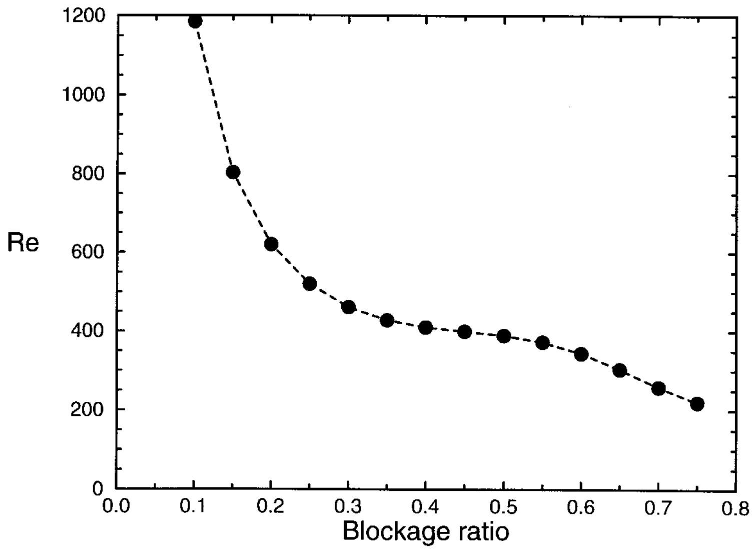

Figure 4. Critical Reynolds number, Re vs. BR for the flow past a sphere with a downstream sting.

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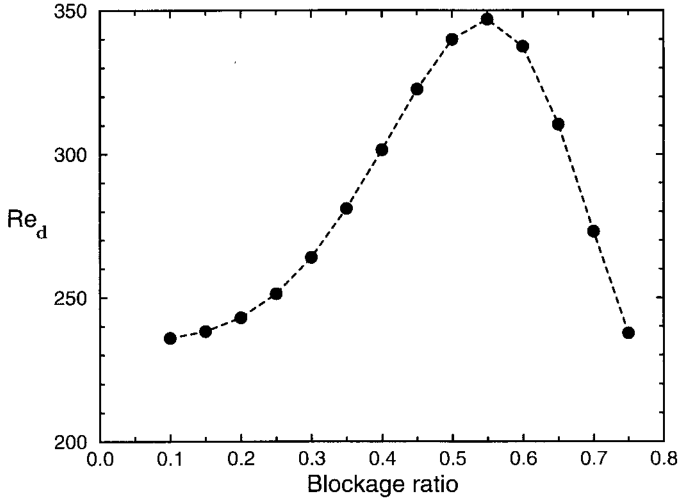

Figure 5. Critical Reynolds number, Re_d vs. BR for the flow past a sphere with a downstream sting.

Table V. Convergence study for BR 0.5.

Number of elements	Critical Re_d	d/b_λ
275	356.7	−71.63
1100	344.6	−43.38
4400	339.9	−43.51

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